Attitude Dynamics of a Rotating Chain of Rigid Bodies in a Gravitational Field

Hari B. Hablani*
Rockwell International, Seal Beach, California

This paper documents in scalar detail a minimum dimension set of discrete coordinate equations of motion of a spacecraft with a chain of hinge-connected rigid bodies in a gravitational field. The equations are nonlinear in attitude angles in orbital plane and linear in attitude rates. The derivation procedure is based on Newtonian mechanics and employs the "direct path" technique. Unknown constraint forces at hinges are eliminated analytically. Translational motion of all the bodies is expressed in terms of attitude motion by using a kinematical identity. Symmetry of an associated mass matrix and antisymmetry of gyroscopic matrix are proved. Also, symmetry of a stiffness matrix corresponding to linear range of attitude angles is verified. The motion equations are numerically illustrated for a five-body spacecraft.

I. Introduction

THE objective of this paper is to document in scalar detail a minimum dimension set of discrete coordinate equations of motion of a spacecraft with a chain of hinge-connected rigid bodies in a gravitational field. It is believed that these equations are not available elsewhere and that they can be used in a number of instances; for example, 1) for an initial estimate of control requirements in terms of power and energy for a complex spacecraft such as Space Station, and 2) to design optimal, large-angle maneuvers of a multibody spacecraft in the presence of gravitational torques. Indeed, precisely these research topics motivated the present study. The spacecraft studied travels round a planet in a circular orbit and rotates once per orbit about its mass center. In the early part of the paper, three-dimensional attitude dynamics of the spacecraft is considered; subsequently, it is specialized to attitude dynamics in the orbital plane only. Section II establishes notational conventions and describes idealizations which form the basis of the mathematical model developed herein. In Sec. III, gravitational forces and torques acting on the hinged bodies of the spacecraft are summarized. Since attitude dynamics of the spacecraft is influenced by constraint forces acting at the hinges, these forces are explicitly determined in Sec. IV by considering translational motion of the spacecraft. Rotational equations are derived in Sec. V. Dynamic analysis in Secs. IV and V is based on Newtonian approach. The equations of motion are nonlinear in attitude angles and linear in attitude rates. Symmetry of the associated mass matrix and antisymmetry of gyroscopic matrix are proved in Sec. V. In addition, the stiffness matrix corresponding to a linear range of attitude angles is shown to be symmetric. In Sec. VI planar attitude dynamics of a five-body spacecraft is illustrated. In order to reveal in explicit detail the structure of the scalar equations of motion derived in generic matrix terms in Sec. V, these matrix equations are expanded for an illustrative two-body spacecraft in Sec. VII. The paper is concluded in Sec. VIII by some closing remarks about the present work.

Development of the motion equations in this paper is based on the "direct path" method of Ho[1] and Hughes.[2] Out symbology is mostly from Ref. 2. Since Keat,[3] and Singh and Likins[4] have provided comprehensive literature surveys of the multibody spacecraft dynamics field, contributions of the past will not be reviewed here; however, where appropriate, present work will be related with previous work.

Submitted March 21, 1984; revision submitted July 23, 1984.

*Member of the Technical Staff, Guidance and Controls Group. Member AIAA.

II. Notational Conventions and Idealizations

An idealized mathematical model of the spacecraft under study is shown in Fig. 1. Due to the multibody nature of the problem, different combinations of the bodies are required for analysis. Consequently, eight subscripts will be used with their ranges specified as follows:

$$p,q=0,1,2,\ldots,N-1,N; \qquad i,j=0,1,2,\ldots,N-1$$
$$m,n=1,2,\ldots,N-1,N; \qquad r,s=1,2,\ldots,N-1 \tag{1}$$

For conciseness an integer to which the summation sign Σ refers will be assumed evident from the context. Furthermore, to ease algebraic manipulations, the following two selection functions will be employed.

$$H_1(\ell)\triangleq\begin{cases}1, & \ell\geq 0\\ 0, & \ell<0\end{cases} \quad ; \quad H_2(\ell)\triangleq\begin{cases}0, & \ell\geq 0\\ 1, & \ell<0\end{cases} \tag{2}$$

which are related thus

$$H_1(\ell)=H_2(-\ell-1), \qquad H_1(\ell)+H_2(\ell)=1 \tag{3}$$

Equations (3) economize the labor required in the detailed proof of various equations appearing in the text.

Various idealizations of the spacecraft model shown in Fig. 1 can be described now. The spacecraft consists of $N+1$ rigid bodies, namely, $\mathcal{B}_p$ which are point-connected at hinges O_n. Nominal attitude of the spacecraft is defined by an orbital frame $\mathfrak{F}_t$. The origin of the frame $\mathfrak{F}_t$ is at composite mass center $\oplus$ of the spacecraft. A dextral triad of unit vectors t_1, t_2, t_3, associated with $\mathfrak{F}_t$ is such that t_1 is along a local vertical, t_2 is along orbital velocity, and $t_3=t_1\times t_2$. The orbital angular velocity of the spacecraft is $\boldsymbol{\Omega}=\Omega t_3$. The main body $\mathcal{B}_0$, which may represent, for example, the NASA Space Shuttle, oscillates with an angular velocity $\boldsymbol{\Omega}_0$ about its mass center $\oplus_0$ relative to the frame $\mathfrak{F}_t$. The remaining bodies $\mathcal{B}_n$ oscillate with angular velocities $\boldsymbol{\Omega}_n$ about the hinges O_n relative to inboard bodies $\mathcal{B}_{n-1}$. The rates $\boldsymbol{\Omega}_p$ are treated to be sufficiently smaller than the orbital rate $\boldsymbol{\Omega}$ so that a nonlinear kinematic term such as $\boldsymbol{\Omega}_p\times\boldsymbol{\Omega}_q$ is ignorable compared to a term $\boldsymbol{\Omega}_p\times\boldsymbol{\Omega}$.

III. Gravitational Forces and Torques Acting on the Spacecraft

Since gravitational influence on a multibody spacecraft has been considered by many investigators in the past (Ho[5] and

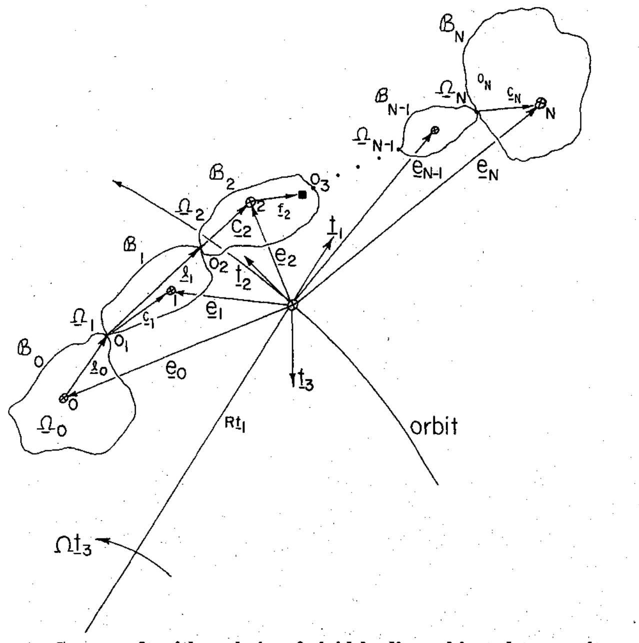

Fig. 1 Spacecraft with a chain of rigid bodies subjected to gravitational forces.

Hooker,[6] for example), details in this section are minimal and are given for completeness only.

Let R be the radius of the orbit of the spacecraft and let e_p denote a vector from the composite mass center $\oplus$ to the mass center $\oplus_p$ of a body $\mathcal{B}_p$. Then, from basic principles, the gravitational force F_G^p experienced by the body $\mathcal{B}_p$ is found to be

$$F_G^p \approx -\Omega^2 m_p \{ R t_1 + (e_p - 3 e_p \cdot t_1 t_1) \} \tag{4}$$

where m_p is the mass of the body $\mathcal{B}_p$. Since the spacecraft is assumed to be in a perfectly circular orbit, it follows that

$$\Sigma m_p e_p = 0 \tag{5}$$

Total gravitational force experienced by the spacecraft is obtained by summing Eq. (4) for all $\mathcal{B}_p$'s and by applying Eq. (5). There follows

$$\Sigma F_G^p \approx -\Omega^2 m R t_1 \tag{6}$$

where m is the mass of the entire spacecraft.

The gravitational torque M_G^p about the mass center $\oplus_p$ of the body $\mathcal{B}_p$ equals

$$M_G^p = 3\Omega^2 [-m_p (e_p \cdot t_1)(e_p \times t_1) + t_1 \times I_p^{\oplus} \cdot t_1] \tag{7}$$

where $I_p^{\oplus}$ is the inertia dyadic of the body $\mathcal{B}_p$ about the composite mass center $\oplus$. An alternate and much simpler expression for M_G^p is obtained by replacing $I_p^{\oplus}$ in terms of central inertia dyadic I_p of the body $\mathcal{B}_p$. It is

$$M_G^p = 3\Omega^2 t_1 \times I_p \cdot t_1 \tag{8}$$

For the outboard bodies $\mathcal{B}_n$ it is desirable to know the gravitational torques G_g^n acting about the hinges O_n. Since

$$G_g^n = c_n \times F_G^n + M_G^n \tag{9}$$

where c_n is a vector from a hinge O_n to a mass center $\oplus_n$, we obtain

$$G_g^n = 3\Omega^2 t_1 \times I_n \cdot t_1 - \Omega^2 m_n c_n \times \{ R t_1 + (e_n - 3 e_n \cdot t_1 t_1) \} \tag{10}$$

The above expressions of force and torque will be used in the following development.

IV. Translational Equations of Motion

Translational motion per se of the spacecraft is not of importance in this paper. However, in order to determine constraint forces at the hinges O_n it needs to be considered. This motion is purely due to attitude motion of the spacecraft, and the two motions are related through spacecraft geometry.

Translational motion of all the bodies $\mathcal{B}_p$ is expressed by the vectors e_p which are constrained by Eq. (5). The vectors e_n, however, can be expressed in terms of the vector e_0 and the system geometry; namely,

$$e_n = e_0 + \ell_0 + \cdots + \ell_{n-1} + c_n \tag{11}$$

The vectors in Eq. (11) are all expressed in different frames, but this point will be treated in Sec. V. The vectors ℓ_i are explained in Fig. 1. Equation (11) can be shortened by introducing the notations

$$\ell_{0p} \triangleq \Sigma H_2 (i-p) \ell_i, \qquad c_{0n} \triangleq \ell_{0n} + c_n \tag{12}$$

Since the vector ℓ_i is zero for $i<0$, $\ell_{00}=0$. The notations [Eq. (12)] contract Eq. (11) to

$$e_n = e_0 + c_{0n} \tag{13}$$

Setting Eq. (13) into Eq. (5) the vector e_0 is found to be

$$e_0(t) = -\Sigma (m_n/m) c_{0n}(t) \tag{14}$$

Attitude motion of all the bodies of the spacecraft is inherent in the vector $c_{0n}(t)$, where t is the independent variable, time. Thus, if the attitudes of all the bodies are known, the vectors e_p will be known through Eqs (13) and (14). These two equations are analog of a "critical kinematical identity" employed by Likins[7] [see Eq. (4) of Ref. 7] in order to derive identical equations of motion from four different procedures.

For an analytic determination of the constraint forces, we will first construct the translational motion equations for each body and then for nests of bodies. A Newtonian derivation procedure will be employed here.

Let a_{cp} be the absolute acceleration of the mass center $\oplus_p$. The hinge force impressed by the body $\mathcal{B}_n$ on the body $\mathcal{B}_{n-1}$ at the hinge O_n is denoted $F_H^{n,n-1}$. Also, F_E^0 is an external force acting on the mass center $\oplus_0$ of the body $\mathcal{B}_0$, and F_E^n is an external force acting on the body $\mathcal{B}_n$ at the hinge O_n. Note that $\Sigma F_E^p = O$ in order to validate the assumption that the composite mass center $\oplus$ moves in a circular orbit. The following translational motion equations can now be written by inspecting free-body diagrams of all the bodies

$$\mathcal{B}_0 : m_0 a_{c0} = F_H^{10} + F_G^0 + F_E^0 \tag{15a}$$

$$\mathcal{B}_s : m_s a_{cs} = -F_H^{s,s-1} + F_H^{s+1,s} + F_G^s + F_E^s \tag{15b}$$

$$\mathcal{B}_N : m_n a_{cN} = F_H^{N-1,N} + F_G^N + F_E^N \tag{15c}$$

where Newton's third law of motion has been used; that is, for example, $F_H^{s-1,s} = -F_H^{s,s-1}$.

We will now derive the expressions for absolute accelerations a_{cp}. For the main body $\mathcal{B}_0$ it is straightforward to show that

$$a_{c0} = -R\Omega^2 t_1 + \ddot{e}_0 \tag{16}$$

where, as usual, $(\dot{\ }) \triangleq$ the inertial time derivative of ( ). Since e_0 is expressed in the orbital frame $\mathcal{F}_t$ which rotates with an angular velocity of Ωt_3, we have

$$\ddot{e}_0 = \mathring{\mathring{e}}_0 + 2\Omega t_3 + \mathring{e}_0 + \Omega^2 (e_0 \cdot t_3 t_3 - e_0) \tag{17}$$

where $(\mathring{\ })$ is the time derivative of the vector ( ) in a rotating frame in which it is expressed. The absolute acceleration a_{c0} is thus determined.

To determine the inertial acceleration a_{cn} we proceed as follows. Inertial location of the mass center $\oplus_n$ is given by Rt_1+e_n. With the aid of Eqs. (11-13) one arrives at

$$a_{cn}=-R\Omega^2 t_1+\ddot{e}_0+\Sigma H_2(p-n)\ddot{\ell}_p+\ddot{c}_n \tag{18}$$

To derive $\ddot{\ell}_i$ and $\ddot{c}_n$ we introduce a frame $\mathfrak{F}_p$ attached with the body $\mathcal{B}_p$; the origin of $\mathfrak{F}_0$ is at the mass center $\oplus_0$ and of $\mathfrak{F}_n$ at the hinge O_n. The frame $\mathfrak{F}_0$ oscillates at an angular velocity of Ω_0 relative to the frame $\mathfrak{F}_t$; the frame $\mathfrak{F}_n$, at Ω_n relative to the frame $\mathfrak{F}_{n-1}$. The vectors ℓ_i and c_n are fixed, respectively, in the frames $\mathfrak{F}_i$ and $\mathfrak{F}_n$. With these definitions the angular velocity ${}^t\omega^p$ of the body $\mathcal{B}_p$ relative to the frame $\mathfrak{F}_t$ can be expressed as

$$^t\omega^p \triangleq \Sigma H_1(p-q)\Omega_q \tag{19}$$

and one can show that

$$\ddot{\ell}_i={}^t\dot{\omega}^i\times\ell_i+2\Omega[t_3\cdot\ell_i{}^t\omega^i-{}^t\omega^i\cdot t_3\ell_i]+\Omega^2[\ell_i\cdot t_3t_3-\ell_i]$$

$$\ddot{c}_n={}^t\dot{\omega}^n\times c_n+2\Omega[t_3\cdot c_n{}^t\omega^n-{}^t\omega^n\cdot t_3c_n]+\Omega^2[c_n\cdot t_3t_3-c_n] \tag{20}$$

The inertial acceleration a_{cn} is obtained by substituting Eqs. (17) and (20) in Eq. (18).

The translational equation of motion for the body $\mathcal{B}_0$ is obtained by substituting Eqs. (16), (17), and (4) for $p=0$ in Eq. (15a). This yields

$$m_0[\overset{\circ\circ}{e}_0+2\Omega t_3\times\overset{\circ}{e}_0+\Omega^2)(e_0\cdot t_3t_3-3e_0\cdot t_1t_1)]=F_H^{10}+F_E^0 \tag{21}$$

Similarly, the motion equation for the body $\mathcal{B}_n$ is obtained by inserting Eqs. (18), (20), and the gravitational force Eq. (4) into Eq. (15b). This leads to

$$\begin{aligned}m_n[\overset{\circ\circ}{e}_0&+\Sigma H_2(p-n){}^t\dot{\omega}^p\times\ell_p+{}^t\dot{\omega}^n\times c_n+2\Omega\{t_3\times\overset{\circ}{e}_0\\&+\Sigma H_2(p-n)(t_3\cdot\ell_p{}^t\omega^p-{}^t\omega^p\cdot t_3\ell_p)+(t_3\cdot c_n{}^t\omega^n-{}^t\omega^n\cdot t_3c_n)\}\\&+\Omega^2e_n\cdot(t_3t_3-3t_1t_1)]=F_H^{n-1,n}+F_H^{n+1,n}+F_E^n\end{aligned} \tag{22}$$

By definition, the hinge force $F_H^{N+1,N}$ in the right side of Eq. (22) for $n=N$ is zero

To evaluate the hinge forces, the following notations which describe geometry of the spacecraft are introduced:

$$\begin{aligned}&\ell_{pn}\triangleq\Sigma H_1(i-p)H_2(i-n)\ell_i; \qquad c_{pn}\triangleq\ell_{pn}+H_1(n-p)c_n;\\&\mu_i\triangleq\Sigma H_2(i-n)m_n; \qquad M_{ip}\triangleq-\Sigma H_2(i-n)m_nc_{pn};\\&\delta_i\triangleq\Sigma H_2(i-n)m_ne_n\end{aligned} \tag{23}$$

The notations ℓ_{pn} and c_{pn} are generalizations of the notations ℓ_{0p} and c_{0n} in Eq. (12). Addition of the last $N-j$ equations associated with the bodies $j+1,\ldots,N$ from Eq. (22) leads to the desired equation of the hinge force

$$\begin{aligned}-F_H^{i+1,i}+\Sigma H_2(i-n)F_E^n&=\mu_i\overset{\circ\circ}{e}_0+\Sigma M_{ip}\times\dot{\Omega}_p+2\Omega\{\mu_it_3\times\overset{\circ}{e}_0\\&-\Sigma(t_3\cdot M_{ip}U-M_{ip}t_3)\cdot\Omega_p\}+\Omega^2\delta_i\cdot(t_3t_3-3t_1t_1)\end{aligned} \tag{24}$$

where U is a unit dyadic.

Through Eq. (14) the vector $e_0(t)$ is known in terms of spacecraft attitude. Consequently, it is desirable to eliminate the derivatives of e_0 from Eq. (24). With the definition of M_{ip} in Eq. (23), Eq. (14) transforms to

$$e_0(t)=(1/m)M_{00} \tag{25}$$

where vector e_0 is expressed in orbital frame $\mathfrak{F}_t$, vector ℓ_i in frame $\mathfrak{F}_i$, and vector c_n in frame $\mathfrak{F}_n$; the vectors ℓ_i and c_n appear in M_{00}. It can be shown that

$$\frac{\mathfrak{F}_{t_d}}{dt}\ell_i=\Sigma H_1(i-p)\Omega_p\times\ell_i,\ \frac{\mathfrak{F}_{t_d}}{dt}\frac{\mathfrak{F}_{t_d}}{dt}\ell_i=\Sigma H_1(i-p)\dot{\Omega}_p\times\ell_i \tag{26}$$

where $\mathfrak{F}_{t_d}/dt\triangleq$ the time derivative in the orbital frame $\mathfrak{F}_t$. Derivatives of c_n similar to Eq. (26) are also determined. Utilizing these derivatives, $\overset{\circ}{e}_0$ and $\overset{\circ\circ}{e}_0$ are found to be

$$\overset{\circ}{e}_0=-\frac{1}{m}\Sigma M_{0p}\times\Omega_p, \qquad \overset{\circ\circ}{e}_0=-\frac{1}{m}\Sigma M_{0p}\times\dot{\Omega}_p \tag{27}$$

Substituting Eq. (27) in Eq. (24) the three-dimensional hinge force takes a simpler form, which is

$$\begin{aligned}-F_H^{i+1,i}+\Sigma H_2(i-n)F_E^n&=\Sigma M'_{ip}\times\dot{\Omega}_p\\&+2\Omega\Sigma(M'_{ip}t_3-t_3\cdot M'_{ip}U)\cdot\Omega_p+\Omega^2\delta_i\cdot(t_3t_3-3t_1t_1)\end{aligned} \tag{28}$$

where

$$M'_{ip}\triangleq M_{ip}-\frac{1}{m}\mu_iM_{0p} \tag{29}$$

So far we have concentrated on three-dimensional motion of the spacecraft. To specialize the model to the dynamics in the orbital plane and to decouple pitch motion from roll-yaw motion, the following simplifications are recognized

$$\Omega_p=\Omega_pt_3,\ t_3\cdot c_{pn}=0, \qquad t_3\cdot e_n=0, \qquad \dot{\Omega}_p=\dot{\Omega}_pt_3 \tag{30}$$

which reduces Eq. (28) to

$$-F_H^{i+1,i}+\Sigma H_2(i-n)F_E^n=\Sigma\dot{\Omega}_pM'_{ip}\times t_3+2\Omega\Sigma\Omega_pM'_{ip}-3\Omega^2\delta_i\cdot t_1t_1 \tag{31}$$

where the hinge force $F_H^{i+1,i}$ is now a two-dimensional vector lying wholly in the orbital plane.

V. Rotational Equations of Motion

First, equations governing the rotational motion of each body of the spacecraft will be derived by treating the hinge forces unknown. Subsequently, these forces will be eliminated by using Eq. (28) or (31) as the case may be.

Drawing a free-body diagram of the body $\mathcal{B}_0$ and applying the Newton-Euler law of motion, its rotation about the mass center $\oplus_0$ is found to be governed by

$$\begin{aligned}I_0\cdot\dot{\Omega}_0&+\Omega[I_0\cdot(t_3\times\Omega_0)+t_3\times I_0\cdot\Omega_0+\Omega_0\times I_0\cdot t_3]\\&+\Omega^2(t_3\times I_0\cdot t_3-3t_1\times I_0\cdot t_1)=G_H^{10}+\ell_0\times F_H^{10}+G_E^0\end{aligned} \tag{32}$$

where the gravitational torque, Eq. (9), has been employed. The hinge torque G_H^{01} is a vector combination of a control torque and a constraint torque, the two being orthogonal to each other; the control torque acts about the free axes of the hinge O_1, and the unknown constraint torque acts about the locked axes. The fact that the constraint torque is unknown is insignificant because, as is now known, it is easily eliminated by taking a dot product of the corresponding equation of motion with the unit vectors along the free axes of the hinge. The control torque portion of the hinge torque G_H^{01} acts on the body $\mathcal{B}_1$ by a motor each of whose parts belongs either to $\mathcal{B}_0$ or to $\mathcal{B}_1$, and $G_H^{01}=-G_H^{10}$. These remarks are applicable to any other hinge torque such as $G_H^{s+1,s}$ to be seen momentarily. In Eq. (32) G_E^0 is an external torque acting on the body $\mathcal{B}_0$ about the mass center $\oplus_0$, and $\ell_0\times F_H^{10}$ is a torque due to the constraint force F_H^{10} at the hinge 0_1.

The free-body diagram of a body $\mathcal{B}_s$ is shown in Fig. 2. $G_H^{s+1,s}$ is the hinge torque exerted by the body $\mathcal{B}_{s+1}$ on the body $\mathcal{B}_s$, positive as shown. A force F_E^s and a torque G_E^s act

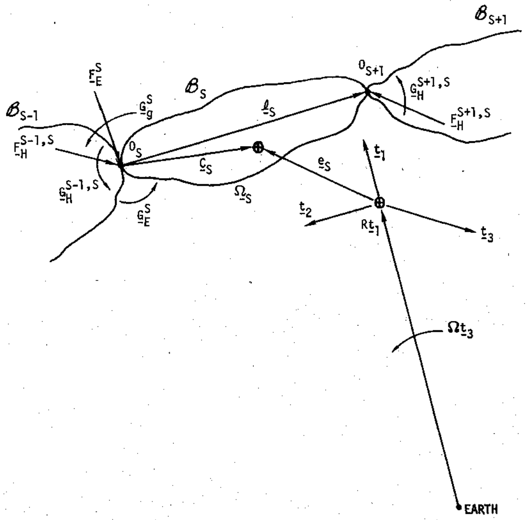

Fig. 2 Free-body-diagram of body $\mathcal{B}_s$.

on the body $\mathcal{B}_s$ at the hinge O_s. They are due to nongravitational force field acting on $\mathcal{B}_s$. G_g^s, given by Eq. (10), is the gravitational torque acting on the body $\mathcal{B}_s$ at the hinge O_s. The rotation of the body $\mathcal{B}_n$ about the hinge O_n is governed by the equation

$$J_n\cdot {}^A\dot{\omega}^n + {}^A\omega^n\times J\cdot {}^A\omega^n = m_n\dot{V}_n\times c_n + T_n \tag{33}$$

where J_n is the inertia dyadic of the body $\mathcal{B}_n$ at the hinge O_n, ${}^A\omega^n$ denotes angular velocity of the body $\mathcal{B}_n$ relative to a Newtonian frame $\mathfrak{F}_A$, $\dot{V}_n$ is inertial acceleration of the hinge O_n, and T_n is the resultant of all "active" torques about the hinge O_n. By applying Eq. (33), a three-dimensional attitude equation for the body $\mathcal{B}_n$ is developed. However, this equation turns out to be unwieldy. To make progress the three-dimensional equation is planarized by invoking the simplifications, Eq. (30) leading to

$$\begin{aligned}
&m_nc_n\times \overset{\circ\circ}{e}_0 + 2\Omega m_n t_3 c_n\cdot\overset{\circ}{e}_0 + \Omega m_n c_n\times\Sigma\Sigma\{H_1(n-p)H_2(q-p)\\
&\quad -H_1(q-p)H_2(q-n)-H_1(n-p)H_2(q-n)\}\ell_p\Omega_p\\
&\quad +\Sigma\{m_n(\ell_{pn}\cdot c_n)t_3 + J_n\cdot t_3 H_1(n-p)\}\dot{\Omega}_p + 3\Omega^2\{-t_1\times J_n\cdot t_1\\
&\quad +m_n(c_n\cdot t_2)(e_{0n}\cdot t_1)t_3\} = G_H^{n-1,n}+G_H^{n+1,n}+G_E^n+\ell_n\times F_H^{n+1,n}
\end{aligned}\tag{34}$$

The torque $G_H^{N+1,N}$ and the force $F_H^{N+1,N}$ in Eq. (34) for $n=N$ are, by definition, zero.

The hinge forces in Eqs. (32) and (34) are eliminated by using Eq. (31). To formulate an equation for Ω_0, Eq. (32), and Eq. (34) for all values of n are added. This operation eliminates all the hinge control torques $G_H^{n-1,n}$. To write the equation compactly, the following symbols are introduced

$$\begin{aligned}
I_{pq} &\triangleq H_1(-p)H_1(-q)I_0 + \Sigma H_1(n-p)H_1(n-q)I_n\\
&\quad +\Sigma m_n(c_{pn}\cdot c_{qn}U - c_{pn}c_{qn})\\
\lambda_{pq}t_3 &\triangleq -\Sigma m_n[H_1(n-p)\{H_1(n-q)+1\}c_n\\
&\quad \times\ell_{qn} + 2\ell_{pn}\times c_{qn}]
\end{aligned}\tag{35}$$

Then the equation for $\Omega_0=\Omega_0 t_3$ is found to be

$$\begin{aligned}
&-M_{00}\times\overset{\circ\circ}{e}_0 - 2\Omega t_3 M_{00}\cdot\overset{\circ}{e}_0 + \Sigma I_{p0}\cdot t_3\dot{\Omega}_p + \Omega\Sigma\lambda_{0p}t_3\Omega_p\\
&\quad +3\Omega^2 t_1\cdot(I_{00}+e_0M_{00})\times t_1 = G_{ET}
\end{aligned}\tag{36}$$

where G_{ET} is the total external torque acting on the spacecraft; that is,

$$G_{ET}=\Sigma G_E^p + \Sigma\ell_{0n}\times F_E^n \tag{37}$$

The oscillations Ω_0 of the main body may be controlled with the torque G_E^0. The equation governing the planar angular velocity $\Omega_n t_3$ is obtained by adding Eq. (34) for the bodies $\mathcal{B}_n$, $\mathcal{B}_{n+1},\ldots,\mathcal{B}_N$ and by eliminating the hinge forces. By adopting this procedure one obtains

$$\begin{aligned}
&-M_{0m}\times\overset{\circ\circ}{e}_0 - 2\Omega t_3 M_{0m}\cdot\overset{\circ}{e}_0 + \Sigma I_{pm}\cdot t_3\dot{\Omega}_p + \Omega\Sigma\lambda_{mp}t_3\Omega_p\\
&\quad +3\Omega^2 t_1\cdot(I_{0m}+e_0M_{0m})\times t_1 = G_{ET}^m + G_H^{m-1,m}
\end{aligned}\tag{38}$$

where

$$G_{ET}^m = \Sigma H_1(n-m)G_E^n + \Sigma\ell_{mn}\times F_E^n \tag{39}$$

The hinge torque $G_H^{m-1,m}$ is a control torque about the axis t_3. The resemblance between Eqs. (36) and (38) is noteworthy.

The vectors e_0, $\overset{\circ}{e}_0$, $\overset{\circ\circ}{e}_0$ in Eqs. (36) and (38) are eliminated by employing Eqs. (25) and (27), transforming the rotational equations to

$$\begin{aligned}
&\mathcal{B}_0:\ \Sigma I'_{p0}\cdot\dot{\Omega}_p t_3 + \Omega\Sigma\lambda_{0p}'\Omega_p t_3 + 3\Omega^2 t_1\cdot I_{00}''\times t_1 = G_{ET}\\
&\mathcal{B}_m:\ \Sigma I_{pm}'\cdot\dot{\Omega}_p t_3 + \Omega\Sigma\lambda_{mp}'\Omega_p t_3 + 3\Omega^2 t_1\cdot I_{0m}''\times t_1 = G_{ET}^m + G_H^{m-1,m}
\end{aligned}\tag{40}$$

where

$$\begin{aligned}
&I_{pq}'\triangleq I_{pq}-\frac{1}{m}(M_{0p}\cdot M_{0q})U,\quad I_{0q}''\triangleq I_{0q}+\frac{1}{m}M_{00}M_{0q},\\
&\lambda_{qp}'\triangleq\frac{2}{m}M_{0q}\cdot(M_{0p}\times t_3)+\lambda_{qp}
\end{aligned}\tag{41}$$

It is known that the mass and stiffness matrices associated with Eq. (40) must be symmetric, whereas the gyroscopic matrix must be antisymmetric. This will be examined now. Inspection of the dyadic I_{pq} in Eq. (35) and I_{pq}' in Eq. (41) reveals their symmetry in the subscript integers p and q because

$$I_{pq}=I_{qp},\qquad I_{pq}'=I_{qp}' \tag{42}$$

With regard to the gyroscopic terms, we first note that for $p=q$

$$\lambda_{pp}=0,\qquad \lambda_{pp}'=0 \tag{43}$$

as required. To prove $\lambda_{pp}=0$, let $p=q$ in λ_{pq} in Eq. (35), substitute c_{pn} from Eqs. (23) and invoke properties (2) and (3); $\lambda_{pp}'=0$ then follows from Eqs. (41). Similarly, it can be proved that

$$\lambda_{pq}t_3=-\lambda_{qp}t_3,\qquad \lambda_{pq}'t_3=-\lambda_{qp}'t_3 \tag{44}$$

Equations (43) and (44) together prove antisymmetry of the gyroscopic terms. Note also that in planar dynamics the terms λ_{pq}, $p\neq q$, depend sinusoidally on attitude of the bodies and, on linearization, $\lambda_{pq}=0$ even for $p\neq q$. The stiffness matrix will be shortly constructed and examined for symmetry.

For computer simulation Eqs. (40) must be written in a matrix form. Dot product of Eqs. (40) with the unit vector t_3 yields

$$\begin{aligned}
&\mathcal{B}_0:\ \Sigma I_{p0}'^{33}\dot{\Omega}_p + \Omega\Sigma\lambda_{0p}'\Omega_p - 3\Omega^2 I_{00}''^{12} = G_{ET}\cdot t_3\\
&\mathcal{B}_m:\ \Sigma I_{pm}'^{33}\dot{\Omega}_p + \Omega\Sigma\lambda_{mp}'\Omega_p - 3\Omega^2 I_{0m}''^{12} = (G_{ET}^m + G_H^{m-1,m})\cdot t_3
\end{aligned}\tag{45}$$

where I'^{33}_{pq} is the (3,3) element of the inertia dyadic I_{pq}' in the orbital frame $\mathfrak{F}_t$, and from Eqs. (35) and (41),

$$I'^{33}_{pq} = I^{33}_{pq} - \frac{1}{m} M_{0p} \cdot M_{0q} \tag{46a}$$

$$I^{33}_{pq} = H_1(-p)H_1(-q)I^{33}_0 + \Sigma H_1(n-p)H_1(n-q)I^{33}_n + \Sigma_{m_n} c_{pn} \cdot c_{qn} \tag{46b}$$

similarly, the term I''^{12}_{0q} is the (1,2) element of the dyadic I''_{0q} of Eq. (41) in the orbital frame $\mathfrak{F}_t$. Equations (45) or (40) are linear in the angular velocity variables Ω_p and nonlinear in attitude angles.

We now introduce attitude angles of the bodies $\mathcal{B}_p$. Let $\phi_0(t)$ be an angle of rotation of the body $\mathcal{B}_0$ about the orbital normal t_3, anticlockwise rotation being positive. Next, let ϕ_n be the attitude of the body $\mathcal{B}_n$ relative to the inboard body $\mathcal{B}_{n-1}$. Clearly, $\Omega_p = \dot{\phi}_p$. The angle of rotation $\phi_{0p}(t)$ of any body $\mathcal{B}_p$ relative to the frame $\mathfrak{F}_t$ will be given by

$$\phi_{0p} \triangleq \phi_0 + \phi_1 + \cdots + \phi_p = \Sigma H_1(p-q)\phi_q \tag{47}$$

whereas the transformation matrix R_{tp} to transform a vector in the frame $\mathfrak{F}_p$ to a vector in the frame $\mathfrak{F}_t$ will be

$$R_{tp} = \begin{bmatrix} \cos\phi_{0p} & -\sin\phi_{0p} & 0 \\ \sin\phi_{0p} & \cos\phi_{0p} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{48}$$

The spacecraft geometry parameters defined in Eqs. (23) can be rewritten in the column matrix form as

$$\ell_{pn} \triangleq \Sigma H_1(i-p)H_2(i-n)R_{ti}\ell_i; \quad c_{pn} = \ell_{pn} + H_1(n-p)R_{tn}c_n \tag{49}$$

where ℓ_i is the column matrix corresponding to the vector ℓ_i in the frame $\mathfrak{F}_i$; the column matrix c_n is defined analogously. The matrix version of the vector M_{ip}, dyadic I_{pq}, and gyroscopic vector $\lambda_{pq}t_3$ will be

$$M_{ip} \triangleq -\Sigma H_2(i-n)m_n c_{pn}$$

$$I_{pq} = H_1(-p)H_1(-q)R_{t0}I_0R_{0t} + \Sigma H_1(n-p)H_1(n-q)R_{tn}I_nR_{nt} + \Sigma m_n[c^T_{pn}c_{qn}U - c_{pn}c^T_{qn}]$$

$$\lambda_{pq} \triangleq -\Sigma m_n\{H_1(n-p)(H_1(n-q)+1)R_{tn}\tilde{c}_nR_{nt}\ell_{qn} + 2\tilde{\ell}_{pn}c_{qn}\}_3 \tag{50}$$

where $R_{pt} = R^T_{tp}$; $U \triangleq$ a 3×3 identity matrix; $\tilde{c}_n$ is the usual 3×3 skew-symmetric matrix corresponding to the column matrix c_n; the matrix $\tilde{\ell}_{pn}$ is defined similarly; $\{\cdot\}_k$ = the t_k component of the column matrix $\{\cdot\}$, $k=1,2,3$; and the superscript "T" means "transpose." The following matrix versions are also required

$$I'_{pq} \triangleq I_{pq} - \frac{1}{m}M^T_{0p}M_{0q}U; \qquad I''_{0q} = I_{0q} + \frac{1}{m}M_{00}M^T_{0q};$$

$$\lambda'_{qp} = \frac{2}{m}M^T_{0q}\begin{bmatrix} \{M_{0p}\}_2 \\ -\{M_{0p}\}_1 \\ 0 \end{bmatrix} + \lambda_{qp} \tag{51}$$

The scalar coefficients appearing in Eqs. (45) can be evaluated now. Specifically, we arrive at

$$I'^{33}_{pq} = I^{33}_{pq} - \frac{1}{m}M^T_{0p}M_{0q}$$

$$I^{33}_{pq} = H_1(-p)H_1(-q)I^{33}_0 + \Sigma H_1(n-p)H_1(n-q)I^{33}_n + \Sigma m_n(c^T_{pn}c_{qn} - [c_{pn}c^T_{qn}]_{33})$$

$$I''^{12}_{0q} = I^{12}_{0q} + \frac{1}{m}\{M_{00}\}_1\{M_{0q}\}_2$$

$$I^{12}_{0q} = H_1(-q)(I^{11}_0 - I^{22}_0)s\phi_{00}c\phi_{00} + \Sigma H_1(n-q)(I^{11}_n - I^{22}_n)s\phi_{0n}c\phi_{0n} - \Sigma m_n\{c_{0n}\}_1\{c_{qn}\}_2 \tag{52}$$

So far it has been assumed that the spacecraft goes through large attitude maneuvers. If the attitude angles are small, a stiffness matrix associated with the terms I''^{12}_{0q} in Eqs. (45) can be identified. The nominal orientation of the spacecraft is the one in which all the attitude angles ϕ_p are zero. For this orientation to be in equilibrium, it is necessary that the components of the vectors ℓ_i and c_n along the vector t_2, when the angles ϕ_p are all zero, must be zero, that is, the mass centers $\oplus_p$'s must all be aligned with local vertical. A stiffness matrix corresponding to this configuration is developed below.

Introduce the approximation that the angles ϕ_p are all small; then

$$R_{tp} \simeq U + \tilde{I}_3\phi_{0p}, \quad \tilde{I}_3 \triangleq \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{53}$$

The geometry parameters ℓ_{pn} and c_{pn} in Eq. (49) are linearized by employing Eq. (53). Similarly, the term I^{12}_{0q} from Eq. (52) and M_{0p} from Eq. (50) are linearized. After linearization it can be shown that

$$-3\Omega^2 I''^{12}_{0q} = +3\Omega^2\Sigma K_{qp}\phi_p \tag{54}$$

where

$$-K_{qp} = H_1(-p)H_1(-q)(I^{11}_0 - I^{22}_0) + \Sigma H_1(n-p)H_1(n-q)(I^{11}_n - I^{22}_n) - \Sigma m_n c_{0n\alpha}d_{npq} + M_{00\alpha}h_{pq} \tag{55a}$$

$$d_{npq} \triangleq \Sigma H_2(i-n)H_1(i-p)H_1(i-q)\ell_{i\alpha} + H_1(n-p)H_1(n-q)c_{n\alpha} \tag{55b}$$

$$h_{pq} \triangleq -\frac{1}{m}\Sigma m_n d_{npq} \tag{55c}$$

$$c_{0n\alpha} \triangleq \Sigma H_2(i-n)\ell_{i\alpha} + c_{n\alpha}; \qquad M_{00\alpha} \triangleq -\Sigma m_n c_{0n\alpha} \tag{55d}$$

In Eqs. (55) $\ell_{i\alpha}$ is a component of the vector ℓ_i in the body-fixed frame $\mathfrak{F}_i$ along an axis which, under equilibrium, is parallel to the local vertical; the component $c_{n\alpha}$ of the vector c_n is defined similarly. It is easily seen that $K_{qp} = K_{pq}$, which proves symmetry of the stiffness matrix. Since in the linear range the gyroscopic terms λ'_{pq} in Eq. (45) are zero, the linear equations that govern planar attitude dynamics of the spacecraft are

$$\mathcal{B}_0: \ \Sigma I'^{33}_{p0}\dot{\Omega}_p + 3\Omega^2\Sigma K_{0p}\phi_p = G_{ET} \tag{56a}$$

$$\mathcal{B}_m: \ \Sigma I'^{33}_{pm}\dot{\Omega}_p + 3\Omega^2\Sigma K_{mp}\phi_p = (G^m_{ET} + G^{m-1,m}_H) \tag{56b}$$

Table 1 Geometry, mass, and inertia parameters of the illustrated spacecraft

Parameters	Rigid body 0	1	2	3	4	Units
$\ell_{i\alpha}$	0.75	1.50	1.50	1.50	–	m
$c_{n\alpha}$	–	0.75	0.75	0.75	2.00	m
m_p	300.00	30.00	30.00	30.00	30.00	kg
I_p^{11}	1000.00	100.00	100.00	250.00	100.00	kg·m^2
I_p^{22}	10000.00	1000.00	1250.00	2000.00	2000.00	kg·m^2
I_p^{33}	12000.00	1200.00	1400.00	2400.00	2400.00	kg·m^2

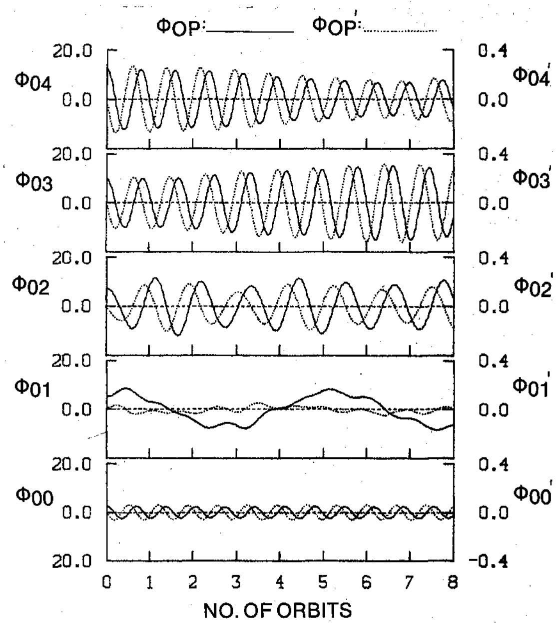

Fig. 3 Natural response of a five-body spacecraft in a gravitational field.

where

$$G_{ET} \triangleq \mathbf{G}_{ET}\cdot \mathbf{t}_3, \quad G_{ET}^m \triangleq \mathbf{G}_{ET}^m\cdot \mathbf{t}_3, \quad G_H^{m-1,m} = \mathbf{G}_m^{m-1,m}\cdot \mathbf{t}_3 \tag{56c}$$

Attitude dependence of the terms $I_{pq}^{\prime 33}$ can be ignored. The linear, matrix, second-order equation corresponding to Eqs. (56) is

$$M\ddot{\phi} + 3\Omega^2 K\phi = G_E + G_H \tag{57a}$$

where M and K are, as usual, the mass and stiffness matrices of the spacecraft, respectively, and

$$\phi(t) \triangleq [\phi_0 \; \phi_1 \; \dots \; \phi_N]^T, \quad G_E(t) \triangleq [G_{ET} \; G_{ET}^1 \; \dots \; G_{ET}^N],$$

$$G_H(t) \triangleq [0 \; G_H^{01} \; \dots \; G_H^{N-1,N}] \tag{57b}$$

The planar attitude dynamics problem treated in this section is analogous to the "transverse deformation case" considered by Likins[8] (see Figs. 8-10 of Ref. 8).

VI. An Illustrative Example

The preceding analysis is illustrated here for a spacecraft having five rigid bodies. The geometry, mass, and inertia parameters of the illustrated spacecraft are shown in Table 1. These parameters are so chosen that the spacecraft is stable; that is, the corresponding stiffness matrix is positive definite. In this example only natural dynamics of the spacecraft is illustrated. The independent variable t is replaced with the orbital angle $\theta \triangleq \Omega t$. The normalized form of the governing equations of motion, Eqs. (45), having time-varying coefficients are integrated numerically. The following initial conditions are used

$$\phi_0 = \phi_1 = \phi_2 = \phi_3 = \phi_4 = 2.5° \quad \phi_0' = \phi_1' = \phi_2' = \phi_3' = \phi_4' = 0.0$$

where $\phi_p' \triangleq \mathrm{d}\phi_p/\mathrm{d}\theta$. The natural response of the spacecraft is shown in Fig. 3. The angle of rotation ϕ_{0p} of each body measured from a local vertical and the rates $\mathrm{d}\phi_{0p}/\mathrm{d}\theta \triangleq \phi_{0p}'$ are displayed against the orbital angle $\theta(t)$. Since any damping or a control torque is not present, the response is periodic. The variation in the amplitudes of oscillations of each body from orbit to orbit manifests interaction between the five bodies. The lowest frequency response of the body $\mathcal{B}_1$, one cycle in four orbits, superimposed by a comparatively high-frequency response of a low-amplitude indicates that the restoring gravity torque on this body is weakest. Note that the angular rates ϕ_{0p}' are much less than unity which conforms with the assumption that the angular rates Ω_p are less than the orbital angular velocity Ω.

The attitude angles in Fig. 3 are in a linear range; therefore, the governing equations are essentially time-invariant. Therefore, it is of interest to know the modal frequencies of the illustrative spacecraft. These are (units: cycles/orbit)

$$\omega_1 = 0.24520 \qquad \omega_2 = 0.92296$$

$$\omega_3 = 1.23010 \qquad \omega_4 = 1.30140$$

$$\omega_5 = 1.45830$$

Although each of these frequencies corresponds to a simultaneous oscillation of all the bodies, examining the response in Fig. 3, loosely speaking, ω_1 corresponds to the oscillations of the body $\mathcal{B}_1$; ω_2, to $\mathcal{B}_2$; ω_3, to $\mathcal{B}_3$; ω_4, to $\mathcal{B}_4$; and ω_5, to $\mathcal{B}_0$.

VII. Usefulness of the Results

As stated in the Introduction, Eqs. (45) and (57) can be used to study attitude maneuver problems. For example, let there be only two bodies, in which case Eqs. (57) reduce to

$$\begin{bmatrix} I_{00}^{\prime 33} & I_{10}^{\prime 33} \\ I_{10}^{\prime 33} & I_{11}^{\prime 33} \end{bmatrix} \begin{bmatrix} \ddot{\phi}_0 \\ \ddot{\phi}_1 \end{bmatrix} + 3\Omega^2 \begin{bmatrix} K_{00} & K_{01} \\ K_{01} & K_{11} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} G_E^0 + G_E^1 \\ G_E^1 + G_H^{01} \end{bmatrix} \tag{58}$$

where

$$I_{00}^{\prime 33} = I_0^{33} + I_1^{33} + m_0 m_1 (\ell_0 + c_1)^2/m$$

$$I_{10}^{\prime 33} = I_1^{33} + m_0 m_1 c_1 (\ell_0 + c_1)/m$$

$$I_{11}^{\prime 33} = I_1^{33} + m_0 m_1 c_1^2/m$$

$$K_{00} = (I_0^{22} - I_0^{11}) + (I_1^{22} - I_1^{11}) + m_0 m_1 (\ell_0 + c_1)\ell_0/m$$

$$K_{01} = K_{11} = (I_1^{22} - I_1^{11}) \tag{59}$$

In Eqs. (59) $\ell_0 \triangleq \ell_{0\alpha}$ and $c_1 \triangleq c_{1\alpha}$. The two attitude angles ϕ_0 and ϕ_1 are controllable via the torques G_E^0 and G_H^{01}, both about the axes parallel to the axis t_3. The central torque G_E^0 acting on the main body may be produced by, say, reaction wheels or jets, whereas the hinge torque G_H^{01} acting on the body $\mathcal{B}_1$ may be generated by a servomechanism. Equation (58) can be employed to determine optimal history of the two control torques to achieve some desired maneuvers. Further remarks, however, are not within the intended scope of this paper.

VIII. Concluding Remarks

A minimum dimension set of discrete coordinate equations of motion of a spacecraft with a chain of hinge-connected rigid bodies in a gravitational field is derived in this paper. In the design of a control system, the gravitational disturbances are generally ignored because the control torques act much faster than the gravitational torques. However, under some circumstances it is desirable to include the gravitational effects explicitly. When such is the case, the equations recorded here in scalar detail will be useful.

Acknowledgments

This study was performed under the cognizance of Mr. Ronald E. Oglevie, Guidance and Controls Group. Partial financial support from Mr. Victor Baddeley, Technical Supervisor, for completing the above efforts is appreciated.

References

[1]Ho, J.Y.L., "Direct Path Method for Flexible Multibody Spacecraft Dynamics," *Journal of Spacecraft and Rockets,* Vol. 14, Feb. 1977, pp. 102-110.

[2]Hughes, P.C., "Dynamics of a Chain of Flexible Bodies," *The Journal of the Astronautical Sciences,* Vol. 27, No. 4, Oct.-Dec. 1979, pp. 359-380.

[3]Keat, J.E., *Dynamical Equations of Multi-body Systems with Application to Space Structure Deployment,* Ph.D. Thesis, Massachusetts Institute of Technology; also, CSDL-T-822, The Charles Stark Draper Laboratory, May 1983.

[4]Singh, R.P. and Likins, P.W., "Manipulator Interactive Design with Interconnected Flexible Elements," *American Control Conference,* 1983.

[5]Ho, J.Y.L., "Remote Manipulator Simulation (RMS)," Lockheed Report LMSC-D403329, Oct. 1974, pp. B15-B18.

[6]Hooker, W., "Equations of Motion for Interconnected Rigid and Elastic Bodies: A Derivation Independent of Angular Momentum," *Celestial Mechanics,* Vol. 11, 1975, pp. 337-359.

[7]Likins, P.W., "Point-Connected Rigid Bodies in a Topological Tree," *Celestial Mechanics,* Vol. 11, 1975, pp. 301-317.

[8]Likins, P.W. and Fleischer, G.E., "Large-Deformation Modal Coordinates for Nonrigid Vehicle Dynamics," Jet Propulsion Laboratory Tech. Rept. 32-1565, Nov. 1972, pp. 45-49.